Theorem (Sums, difference and scalar multiples of Riemann integrable functions are Riemann integrable)

If \( f \) and \( g \) are Riemann integrable on \( [a, b] \), then \( f + g \), \( f - g \), and \( cf \) are Riemann integrable and
\[
1. \quad S_a^b f + g = S_a^b f + S_a^b g \\
2. \quad S_a^b cf = c S_a^b f
\]

Proof

1. Let \( \varepsilon > 0 \) be given.
   There exists \( \delta_1 > 0 \) so that if \( \|P\| < \delta_1 \), then \( |S(P, f) - S_a^b f| < \frac{\varepsilon}{2} \)
   There exists \( \delta_2 > 0 \) so that if \( \|P\| < \delta_2 \), then \( |S(P, g) - S_a^b g| < \frac{\varepsilon}{2} \)

   Let \( \delta = \min\{\delta_1, \delta_2\} \)

   Assume \( \|P\| < \delta \)

   \[
   S(P, f + g) = \sum_{i=1}^{n} (f(x_i^*) + g(x_i^*)) \Delta x_i = \sum_{i=1}^{n} f(x_i^*) \Delta x_i + \sum_{i=1}^{n} g(x_i^*) \Delta x_i = S(P, f) + S(P, g)
   \]

   \[
   |S(P, f + g) - S_a^b f + S_a^b g| = |S(P, f) - S_a^b f + S(P, g) - S_a^b g| \\
   \leq |S(P, f) - S_a^b f| + |S(P, g) - S_a^b g| \\
   < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
   \]

2. If \( c = 0 \), then \( cf = 0 \) is a constant which is Riemann integrable.

   If \( c \neq 0 \) then

   There exists \( \delta > 0 \) so that if \( \|P\| < \delta \), then \( |S(P, f) - S_a^b f| < \frac{\varepsilon}{|c|} \)

   Assume \( \|P\| < \delta \)

   \[
   |S(P, cf) - c S_a^b f| = |c S(P, f) - c S_a^b f| \\
   = |c| |S(P, f) - S_a^b f| \\
   < |c| \delta \\
   < |c| \frac{\varepsilon}{|c|} \\
   = \varepsilon
   \]
Thm (Integration maintains order)
Let \( f, g \) be Riemann integrable with \( f(x) \leq g(x) \) for all \( x \in [a, b] \)***
then \( \int_a^b f \leq \int_a^b g \)

proof

Case 1 Assume \( h(x) \geq 0 \) for all \( x \in [a, b] \)
\[
L(P, h) = \sum_{i=1}^{n} m_i \Delta x_i \geq 0 \quad \text{for any partition } P
\]
\[
\Rightarrow \sup\{L(P, h) : P \in \mathcal{P}\} \geq 0
\]
\[
\Rightarrow \int_a^b h = \int_a^b h \geq 0
\]

Case 2 \( g(x) - f(x) \geq 0 \)
Apply case 1 with \( h(x) = g(x) - f(x) \) and difference them
\[
0 \leq \int_a^b (g - f) = \int_a^b g - \int_a^b f
\]
\[
\Rightarrow \int_a^b f \leq \int_a^b g
\]

Thm (Integrability when you split an interval)
If \( f \) is Riemann Integrable on \([a, b]\) and \( a < c < b \) then \( f \) is Riemann Integrable on \([a, c]\) and \([c, b]\) and \( \int_a^b f = \int_a^c f + \int_c^b f \)

proof

Let \( \varepsilon > 0 \) be given
There exists a partition \( P \) such that \( U(P, f) - L(P, f) < \varepsilon \)
Let \( P_1 = P \cap [a, c] \) then \( P_1 \cup P_2 \) is a partition of \([a, c]\)
Let \( P_2 = P \cap [c, b] \) then \( P_2 \cup P_3 \) is a partition of \([c, b]\)
\[
U(P, f) = U(P_1, f) + U(P_2, f) \quad \text{and} \quad L(P, f) = L(P_1, f) + L(P_2, f)
\]
\[
U(P_1, f) - L(P_1, f) = U(P_1, f) - U(P_2, f) - (L(P_1, f) - L(P_2, f))
\]
\[
= U(P, f) - L(P, f) - (U(P_2, f) - L(P_2, f)) \quad \text{positive}
\]
\[
< \varepsilon
\]
\[ U(P_2,f) - L(P_2,f) = U(P_1,f) - U(P_1,f) - (L(P_1,f) - L(P_1,f)) \]
\[ = U(P, f) - L(P, f) - (U(P, f) - L(P, f)) \text{ for all } n \]
\[ < \epsilon \]

So \( f \) is Riemann integrable on both \([a, c] \) and \([c, b] \) since we found a partition where the difference between upper and lower sums is less than \( \epsilon \).

\[ \int_{a}^{b} f \leq U(P, f) \]
\[ = U(P_1, f) + U(P_2, f) \]
\[ < L(P_1, f) + L(P_2, f) + \epsilon \]
\[ < \int_{a}^{c} f + \int_{c}^{b} f + \epsilon \]

Since \( \epsilon \) is arbitrary,

\[ \int_{a}^{b} f \leq \int_{a}^{c} f + \int_{c}^{b} f \]

On the other hand,

\[ \int_{a}^{b} f \geq L(P, f) \]
\[ = L(P_1, f) + L(P_2, f) \]
\[ > U(P_1, f) + U(P_2, f) \]
\[ > \int_{a}^{c} f + \int_{c}^{b} f - \epsilon \]

Since \( \epsilon \) is arbitrary,

\[ \int_{a}^{b} f \geq \int_{a}^{c} f + \int_{c}^{b} f \]

\[ \therefore \int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f \]

In general, the composition of two Riemann integrable functions is not necessarily Riemann integrable. 

Thm (Composition of Riemann Integrable and Continuous is Riemann Integrable)

If \( f: [a, b] \rightarrow \mathbb{R}, M \) is Riemann Integrable and \( g: \mathbb{R} \rightarrow M \) is continuous, then \( g \circ f \) is Riemann Integrable.

proof (see book)
Thm (Integrability of Product, Absolute Value, min, max)

Let \( f, g \) be Riemann Integrable then
1. \( f g \) is Riemann Integrable
2. \( |f| \) is Riemann Integrable and \( \int_a^b |f| \leq \int_a^b |f| \)
3. \( \text{min}\{f, g\} \) and \( \text{max}\{f, g\} \) are Riemann Integrable

proof (see book)

Thm

If \( f \) is Riemann Integrable and \( q = f \) except at a finite number of points on \([a, b]\) then \( g \) is Riemann Integrable and \( \int_a^b g \)

proof

Case 1 \( h(x) = 0 \) except at a finite number of points on \([a, b]\)

\[
h(x) = \begin{cases} c_i & x_i < x < x_{i+1} \\ 0 & x \notin [x_i, x_{i+1}] \end{cases}
\]

Let \( \delta = \frac{\varepsilon}{C_m} \) and \( C_m = \text{max}\{c_1, c_2, \ldots, c_k\} \)

Let \( \varepsilon > 0 \) be given

Let \( \delta_1 = \delta \)

Let the partition \( P \) be given by:

* If \( x_i - \frac{\varepsilon}{3} < x < x_{i+1} - \frac{\varepsilon}{3} \) if not eliminate \( x_i - \frac{\varepsilon}{3} \)

\[
U(P, h) - L(P, h) \leq \sum_{i=1}^{k} C_M (x_i + \frac{\varepsilon}{3} - (x_i - \frac{\varepsilon}{3})) - \sum_{i=1}^{k} C_m (x_i + \frac{\varepsilon}{3} - (x_i - \frac{\varepsilon}{3}))
\]

\[
= \sum_{i=1}^{k} (C_M - C_m) \frac{2\varepsilon}{3}
\]

\[
< \frac{2k (C_M - C_m) \frac{\varepsilon}{3}}{2k (C_M - C_m)} = \varepsilon
\]

\[
< 2k (C_M - C_m) \frac{\varepsilon}{3}
\]
\[ U(P, h) \leq \frac{2k \cdot C_M \cdot \varepsilon}{3} \leq \frac{C_h}{C_M} \cdot \varepsilon \leq \varepsilon \]

\[ \Rightarrow \sum_{a} h = 0 \]

**Case 2** \[ f(x) = g(x) \] for all \( x \) except \( x_1, x_2, \ldots, x_k \)

Let \( h(x) = f(x) - g(x) \)

\[ h(x) = 0 \text{ for all } x \text{ except } x_1, x_2, \ldots, x_k \]

Apply Case 1

\[ \sum_{a} h = 0 \Rightarrow \sum_{a} f - g = 0 \]

\[ \Rightarrow \sum_{a} f - \sum_{a} g = 0 \]

\[ \Rightarrow \sum_{a} f = \sum_{a} g \]

**Thm (Riemann Integrable on Subintervals)**

If \( f \) is bounded and Riemann Integrable on every subinterval of \([a, b]\) then \( f \) is Riemann Integrable on \([a, b]\).

**Proof** (see book)