Des'n

A Riemann Sum for a bounded function \( f \) on an interval \([a,b]\) with respect to a partition \( P \) is a choice of points: \( t_1, t_2, \ldots, t_n \) with \( x_{i-1} \leq t_i \leq x_i \) and the Riemann sum is given by:

\[
S(P, f) = \sum_{i=1}^{n} f(t_i) \Delta x_i
\]

Lemma

\( L(P, f) \leq S(P, f) \leq U(P, f) \)

proof

\[
\inf \{ f(x) : x \in [x_{i-1}, x_i] \} \leq f(t_i) \leq \sup \{ f(x) : x \in [x_{i-1}, x_i] \}
\]

\[
\min_i \leq f(t_i) \leq \max_i
\]

\[
\sum_{i=1}^{n} \min_i \Delta x_i \leq \sum_{i=1}^{n} f(t_i) \Delta x_i \leq \sum_{i=1}^{n} \max_i \Delta x_i
\]

\( L(P, f) \leq S(P, f) \leq U(P, f) \)

Des'n

The Limit of a Riemann Sum of \( f \) with respect to a partition \( P \) as the mesh of \( P \) goes to zero is \( I \) (denoted \( \lim_{\|P\| \to 0} S(P, f) = I \)) iff

For every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( \|P\| < \delta \) then \( |S(P, f) - I| < \varepsilon \).

Thrm (A function is Riemann Integrable iff the limit of the Riemann Sums)

proof (See book)

The strength of this theorem is that it allows us to choose any point in the subinterval to evaluate to get the value of the function.
Then (Continuity insures Riemann Integrability)

If \( f: [a,b] \rightarrow \mathbb{R} \) is continuous then \( f \) is Riemann Integrable.

**Proof**

If \( f: [a,b] \rightarrow \mathbb{R} \) is continuous then \( f \) is uniformly continuous since \([a,b]\) is a closed bounded interval.

Let \( \varepsilon > 0 \) be given.

Choose \( \delta > 0 \) such that if \( |x - y| < \frac{\varepsilon}{b-a} \), then \( |f(x) - f(y)| < \frac{\varepsilon}{b-a} \).

Let \( P \) be a partition with \( ||P|| < \delta \).

Since \( f \) is continuous on \([a,b]\), \( f \) is continuous on each subinterval \([x_{i-1}, x_i]\) of \( P \). On each subinterval \( f \) attains its maximum and minimum value \( M_i \) and \( m_i \). There exist \( x_{m_i} \in [x_{i-1}, x_i] \) and \( x_{M_i} \in [x_{i-1}, x_i] \) such that \( M_i = f(x_{M_i}) \) and \( m_i = f(x_{m_i}) \).

Since \( |x_i - x_{i-1}| < \delta \) \( \Rightarrow |x_{M_i} - x_{m_i}| < \delta \),

\[ \Rightarrow |f(x_{M_i}) - f(x_{m_i})| < \frac{\varepsilon}{b-a} \]

\[ \Rightarrow M_i - m_i < \frac{\varepsilon}{b-a} \]

\[ U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \]

\[ < \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i \]

\[ < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \]

\[ < \frac{\varepsilon}{b-a} (b-a) \]

\[ \therefore f \text{ is Riemann Integrable}. \]

As \( \varepsilon \) becomes arbitrarily small, so does \( U(P, f) - L(P, f) \), which implies that \( f \) is Riemann Integrable.

The converse is not true.

\[ f(x) = \begin{cases} 1 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases} \text{ is not continuous on } [0, 1] \text{ but is Riemann Integrable.} \]

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**Venn Diagram**
Example

Let \([a, b]\) be a closed bounded interval

Let \(x_0 = a + \frac{c(b-a)}{n}\), \(c = 0, 1, 2, \ldots, n\)

\(\{x_i\}\) is a partition of \([a, b]\)

\begin{align*}
x_0 &= a + \frac{0(b-a)}{n} = a \\
x_{c-1} &< x_c \\
x_n &= a + \frac{n(b-a)}{n} = b
\end{align*}

A Riemann sum is:

\[
\sum_{i=1}^{n} f(x_i) \Delta x_i = \sum_{i=1}^{n} f(x_i) \frac{b-a}{n} = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right)
\]

As \(n \to \infty\) this is the \(S_0\) of \(f\) if \(f\) is Riemann Integrable.

Example

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{n}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} \left( \frac{n(n+1)}{2} \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{2} \left( 1 + \frac{1}{n} \right)
\]

\[
= \frac{1}{2}
\]

Let \(f(x) = x\) and \([a, b] = [0, 1]\)

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}
\]

is a Riemann sum for:

\[
S_0^{1} x
\]

In the last section we proved

\[
S_0^{1} x = \frac{1}{2} \left( 1^2 - 0^2 \right) = \frac{1}{2}
\]
Example

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \cos \left( \frac{\pi i}{2n} \right) \]

Let \( f(x) = \cos(x) \)
and \( [a, b] = [0, \frac{\pi}{2}] \)

\[ x_i = \frac{i}{n} (b - a) = \frac{\pi i}{2n} \]

\[ = \int_{0}^{\pi/2} 2 \cos x \, dx \]

\[ = 2 \sin x \bigg|_{0}^{\pi/2} \]
\[ = 2 \left( \sin \frac{\pi}{2} - \sin 0 \right) \]
\[ = \frac{2}{\pi} \]

\[ = \frac{\pi}{\pi} \]