Definition
A function $f: D \to \mathbb{R}$ is uniformly continuous on $D$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in D$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Example Let $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$
Prove $f$ is uniformly continuous on $\mathbb{R}$

Proof
Let $\epsilon > 0$ be given
choose $\delta < \frac{\epsilon}{3}$
Assume $|x - y| < \delta$

$|f(x) - f(y)| = |3x + 5 - (3y + 5)|$

$= |3x - 3y|$

$= 3|x - y|$

$< 3\delta$

$< \frac{\epsilon}{3}$

$\leq \epsilon \quad \Box$

(In general any line is uniformly continuous)

Example Let $f: [0, 1] \to \mathbb{R}$ given by $f(x) = \frac{1}{x+2}$
Prove $f$ is uniformly continuous

Proof
Let $\epsilon > 0$ be given
choose $\delta < 4\epsilon$
Assume $|x - y| < \delta$

$|f(x) - f(y)| = \left| \frac{1}{x+2} - \frac{1}{y+2} \right|$

$= \left| \frac{y+2-(x+2)}{(x+2)(y+2)} \right|$

$= \frac{|x-y|}{(x+2)(y+2)}$

$\leq \frac{|x-y|}{4} < \frac{\delta}{4} < \frac{4\epsilon}{4} = \epsilon \quad \Box$
A function that is not uniformly continuous is difficult to prove.

**Negation of the def'n of uniformly continuous**

\[ f : D \to \mathbb{R} \text{ is not uniformly continuous iff there exists an exceptional } \varepsilon_0 > 0 \text{ such that for all } \delta > 0 \text{ there exist } x, y \in D \text{ with } |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon_0. \]

**Theorem (Not uniformly continuous criteria)**

A function \( f : D \to \mathbb{R} \) is not uniformly continuous iff there exists sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} |x_n - y_n| = 0 \) and \( \lim_{n \to \infty} |f(x_n) - f(y_n)| \geq \varepsilon_0 > 0 \)

(Note we do not require the sequences to converge.)

**Proof**

\[ \Rightarrow \text{ Let } f \text{ not be uniformly continuous.} \]

There exists \( \varepsilon_0 > 0 \) such that for all \( \delta > 0 \) there exists \( x, y \in D \) with \( |x - y| < \delta \) but \( |f(x) - f(y)| < \varepsilon_0 \).

Let \( \delta_n = \frac{1}{n} > 0 \)

For each \( \delta_n \) there exists \( x_n, y_n \in D \) such that \( |x_n - y_n| < \delta_n \)

\( \text{but } |f(x_n) - f(y_n)| \geq \varepsilon_0 \)

\( 0 < |x_n - y_n| < \frac{1}{n} \)

\( \lim_{n \to \infty} |x_n - y_n| = 0 = \lim_{n \to \infty} \frac{1}{n} = 0 \) by squeeze theorem

\[ \Leftarrow \text{ Assume there exists sequences } (x_n)_{n \in \mathbb{N}} \text{ and } (y_n)_{n \in \mathbb{N}} \text{ in } D \]

and \( \varepsilon_0 > 0 \) such that \( \lim_{n \to \infty} |x_n - y_n| = 0 \) and \( |f(x_n) - f(y_n)| \geq \varepsilon_0 \).

The \( \varepsilon_0 \) is the exceptional \( \varepsilon_0 \).

If \( \delta > 0 \) then there exists \( n_0 \in \mathbb{N} \) such that \( n > n_0 \)

\( \text{then } |x_n - y_n| < \delta \) but \( |f(x_n) - f(y_n)| > \varepsilon_0 \)

Therefore, \( f \) is not uniformly continuous. \( \blacksquare \)
The Domain is important!

Example $f : [-20, 20] \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is uniformly continuous.

Proof Let $\varepsilon > 0$ be given.

Choose $\delta < \frac{\varepsilon}{40}$

Assume $|x-y| < \delta$

$$|f(x) - f(y)| = |x^2 - y^2|$$

$$= |(x-y)(x+y)|$$

$$= |x-y| |x+y|$$

$$\leq \delta |x+y|$$

$$\leq 40 \delta$$

$$< 40 \frac{\varepsilon}{40}$$

$$= \varepsilon$$

$\therefore \delta$ is uniformly continuous

Example $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.

Proof Let $x_n = n$ and $y_n = n + \frac{1}{n}$

$$\lim_{n \to \infty} |x_n - y_n| = \lim_{n \to \infty} |n - (n + \frac{1}{n})| = \lim_{n \to \infty} \frac{1}{n} = 0$$

$$|f(x_n) - f(y_n)| = |n^2 - (n + \frac{1}{n})^2|$$

$$= |n^2 - (n^2 + 2 + \frac{1}{n^2})|$$

$$= |-2 - \frac{1}{n^2}|$$

$$\geq 2$$

$\therefore f$ is not uniformly continuous.

Note: Any uniformly continuous function is continuous. This can be easily prove to $\lim_{x \to y} f(x) = f(c)$ in the definition of continuity. The above example shows the converse is not true.
Theorem: Continuity on a closed bounded interval is uniformly continuous.

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous then \( f \) is uniformly continuous.

**Proof (by contradiction):**

Assume \( f : [a, b] \rightarrow \mathbb{R} \) is continuous and \( f \) is not uniformly continuous.

There exists sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) in \([a, b]\) and \( \varepsilon_0 > 0 \) such that
\[
\lim_{n \to \infty} |x_n - y_n| = 0 \quad \text{and} \quad |f(x_n) - f(y_n)| > \varepsilon_0.
\]

By the Bolzano-Weierstrass theorem \( (x_n)_{n \in \mathbb{N}} \) has a convergent subsequence \((x_{n_k})_{k \in \mathbb{N}}\) say \( \lim_{k \to \infty} x_{n_k} = C \in [a, b] \).

Now
\[
- |x_n - y_n| \leq x_n - y_n < |x_n - y_n|
\]
\[
\lim_{n \to \infty} x_n - y_n = 0 \quad \text{by squeeze theorem}
\]
\[
\lim_{k \to \infty} x_{n_k} - y_{n_k} = 0 \quad \text{subsequences converge to same limit}
\]
\[
\lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} y_{n_k} - x_{n_k} + x_{n_k} = \lim_{k \to \infty} y_{n_k} - x_{n_k} + \lim_{k \to \infty} x_{n_k} = C
\]

The function \( f \) is continuous at \( C \).

Let \( \varepsilon < \varepsilon_0 \) then there is \( \delta > 0 \) so that
\[
\text{if } |x - C| < \delta \quad \text{then} \quad |f(x) - f(c)| < \varepsilon
\]

Choose \( k_1 \) so that if \( k > k_1 \) \( |x_{n_k} - C| < \frac{\delta}{2} \)

Choose \( k_2 \) so that if \( k > k_2 \) \( |y_{n_k} - C| < \frac{\delta}{2} \)

Let \( k_0 = \max\{k_1, k_2\} \)

\[
|f(x_{n_k}) - f(y_{n_k})| = |f(x_{n_k}) - f(C) + f(C) - f(y_{n_k})|
\]
\[
\leq |f(x_{n_k}) - f(C)| + |f(C) - f(y_{n_k})|
\]
\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\]
\[
= \varepsilon
\]

But
\[
|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 > 0 \quad \text{contradiction}
\]

\( f \) must be uniformly continuous.
The uniformly continuous image of a Cauchy sequence is a Cauchy sequence.

Let \( f: \mathbb{D} \to \mathbb{R} \) be uniformly continuous and \((x_n)_{n=1}^\infty\) a Cauchy sequence then \((f(x_n))_{n=1}^\infty\) is a Cauchy sequence.

**Proof**

Let \( \varepsilon > 0 \) be given.

Since \( f \) is uniformly continuous there is a \( \delta > 0 \) such that if \( |x - y| < \delta \) then \( |f(x) - f(y)| < \varepsilon \).

Since \((x_n)_{n=1}^\infty\) is Cauchy there is no such that if \( n \geq N \)

then \( |x_n - x_m| < \delta \).

Assume \( n > m \)

then \( |x_n - x_m| < \delta \) \( \implies \) \( |f(x_n) - f(x_m)| < \varepsilon \).

\( \therefore \) \((f(x_n))_{n=1}^\infty\) is a Cauchy sequence.

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**Theorem** (The Converse is True)

If \( f: \mathbb{D} \to \mathbb{R} \) has the property that the image of a Cauchy sequence is a Cauchy sequence then \( f \) is uniformly continuous.

**Proof** (See Book)

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**Theorem** (Uniformly continuous function are extendable)

Let \( f: (a, b) \to \mathbb{R} \) be uniformly continuous then \( f \) has a continuous extension to the closed interval \([a, b]\) (i.e., there is \( g: [a, b] \to \mathbb{R} \) continuous such that \( g|_{(a, b)} = f \)).

**Proof** (See Book)