Theorem (Continuous functions on closed bounded intervals are bounded)

If \( f: [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), then \( f \) is bounded on \([a, b]\).

Proof (by contradiction)

Suppose \( f \) is not bounded on \([a, b]\).
There exists \( x_1 \in [a, b] \) such that \( |f(x_1)| > 1 \).
There exists \( x_2 \in [a, b] \) such that \( |f(x_2)| > 2 \).

...;

There exists \( x_n \in [a, b] \) such that \( |f(x_n)| > n \).

We construct a sequence \( (x_n)_{n \in \mathbb{N}} \) of \([a, b]\) such that \( |f(x_n)| > n \).

By the Bolzano-Weierstrass theorem there exists a convergent subsequence \( (x_{n_k})_{k \in \mathbb{N}} \);

\( (x_{n_k})_{k \in \mathbb{N}} \) converges to say \( x \in [a, b] \).

Since \( f \) is continuous the sequence \( (f(x_{n_k}))_{k \in \mathbb{N}} \) is also convergent.

\( (f(x_{n_k}))_{k \in \mathbb{N}} \) is bounded, but \( |f(x_{n_k})| > n_k \geq k \)

which is not bounded (contradiction).

\( \therefore \) \( f \) must be bounded on \([a, b]\).

Definition

A function \( f: D \to \mathbb{R} \) has an absolute maximum if there exists \( x_0 \in D \) such that \( f(x) \leq f(x_0) \) for all \( x \in D \).

A function \( f: D \to \mathbb{R} \) has an absolute minimum if there exists \( x_0 \in D \) such that \( f(x_0) \leq f(x) \) for all \( x \in D \).
Thm (Continuous functions on a closed bounded interval attaining maximum and minimum)
If \( f: [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), then \( f \) has an absolute maximum and minimum on \([a, b]\).

**Proof**

Let \( \alpha = \inf \{ f(x) : x \in [a, b] \} \) and \( \beta = \sup \{ f(x) : x \in [a, b] \} \).

For each \( n \in \mathbb{N} \) there exist \( x_n \in [a, b] \) and \( y_n \in [a, b] \) such that:

\[
\alpha \leq f(x_n) < \alpha + \frac{1}{n} \quad \text{and} \quad \beta - \frac{1}{n} < f(y_n) \leq \beta.
\]

By the Bolzano-Weierstrass Theorem there exists convergent subsequences \((x_{n_k})_{k \in \mathbb{N}}\) and \((y_{n_j})_{j \in \mathbb{N}}\).

Say, \( \lim_{k \to \infty} x_{n_k} = x_0 \in [a, b] \) and \( \lim_{j \to \infty} y_{n_j} = y_0 \in [a, b] \).

Since \( f \) is continuous,

\[
\lim_{k \to \infty} f(x_{n_k}) = f(x_0) \quad \text{and} \quad \lim_{j \to \infty} f(y_{n_j}) = f(y_0).
\]

By the squeeze theorem,

\[
\lim_{k \to \infty} \alpha \leq \lim_{k \to \infty} f(x_{n_k}) \leq \lim_{k \to \infty} \alpha + \frac{1}{n_k} \quad \text{and} \quad \lim_{j \to \infty} \beta - \frac{1}{n_j} \leq \lim_{j \to \infty} f(y_{n_j}) \leq \lim_{j \to \infty} \beta.
\]

\[
\alpha \leq f(x_0) \leq \alpha \quad \text{and} \quad \beta \leq f(y_0) \leq \beta.
\]

\[
\Rightarrow f(x_0) = \alpha \quad \text{and} \quad f(y_0) = \beta.
\]

The value of \( \alpha \) is a lower bound and the value of \( \beta \) is an upper bound.

\[
\Rightarrow f \text{ has a absolute maximum and minimum on } [a, b].
\]

**Example** \( f: (0, 1] \to \mathbb{R} \) by \( f(x) = \frac{1}{x} \)

\( f \) is continuous on \((0, 1]\), but \( f \) has no maximum.
Let \( f \) be a function where \( f(c) = k \).

**Claim:**

Assume \( f(c) = k \).

**Case 1:** \( f(c) = k \).

Let \( \epsilon = \frac{1}{2} \).

Since \( f \) is continuous at \( c \), there exists \( \delta > 0 \) such that if \( 0 < |x - c| < \delta \) then \( |f(x) - f(c)| < \frac{1}{2} \).

Let \( S = \{ x \in [a, b] : f(x) < k \} \).

Since \( f(a) < k \), \( a \in S \).

Let \( c = \sup S \) such that \( c \leq b \).

Now \( f(c) = \inf f([a, b]) \).

Since \( f(c) < k \), \( f(c) < f(b) \).

Choose \( \epsilon > 0 \) such that \( f(c) < f(b) + \epsilon \).

Let \( \delta = \min( \delta, \frac{1}{2} ) \).

Since \( f \) is continuous at \( c \), there exists \( \delta > 0 \) such that if \( 0 < |x - c| < \delta \) then \( |f(x) - f(c)| < \epsilon \).

**Proof:**

Let \( \inf f(C) = k \).

Then for every \( k \) there exists \( c \in [a, b] \) such that \( f(c) = k \).

Let \( f(x) \to k \) as \( x \to a \).

Let \( x = \frac{1}{n} \) for \( n \to \infty \).

Then \( f(x) \to 0 \).

By contradiction, \( g(x) = \frac{1}{x-1} \).

Then \( g(x) \to \infty \) as \( x \to 1 \).

The point of both examples shows both the continuous and the domain being a closed bounded interval are needed.
Case 2 $f(c) > k$

Let $0 < \varepsilon < f(c) - k \implies k < f(c) - \varepsilon$

Since $f$ is continuous at $c$ there exists $\delta > 0$ such that

if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

$\implies -\varepsilon < f(x) - f(c) < \varepsilon$

$\implies k < f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

Let $x_0 \in (c - \delta, c)$ and $x_0 \in [a, b]$

then $|x_0 - c| < \delta$ and $k < f(c) - \varepsilon < f(x_0)$

For all $x \in (x_0, c)$ $k < f(x)$ or $\inf S > k$

since each $x < c$ and $c$ is a contradiction

$\implies Sup S \leq x_0$

$\implies sup S \neq c$ and $c$ is the contradiction.

Case 2 is not possible.

\[ f(c) = k \]

Case 2 $f(b) < f(a) \implies f(b) < k < f(a)$

$\implies -f(a) < -k < -f(b)$

Apply case 1 to $-f$ to obtain $c \in [a, b]$ such that $-f(c) = -k$

$\implies f(c) = k$

There always exists $c \in [a, b]$ such that $f(c) = k$.

Corollary

Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a)f(b) < 0$. Then there exist $c \in [a, b]$ such that $f(c) = 0$.

Proof.
$f(a)f(b) < 0$

$\implies f(a)$ and $f(b)$ are of different sign

$\implies f(a) \neq f(b)$

Let $k = 0$ and apply Intermediate Value Theorem.