Thm \hspace{1em} (Sequential Continuity is Equivalent to Continuity)

Let \( f : D \rightarrow \mathbb{R} \) and \( c \in D \). Then \( f \) is continuous at \( c \) iff for every sequence \( (x_n)_{n \in \mathbb{N}} \) in \( D \) with \( \lim_{n \to \infty} x_n = c \) then \( \lim_{n \to \infty} f(x_n) = f(c) \).

**Proof**

\[ \exists \] Assume \( f \) is continuous at \( c \).

Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( D \) with \( \lim_{n \to \infty} x_n = c \).

Let \( \varepsilon > 0 \) be given.

Because \( f \) is continuous at \( c \) there exist \( \delta > 0 \) so that \( |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \).

For this choice of \( \delta > 0 \) there exist \( n_0 \in \mathbb{N} \) so that \( n > n_0 \Rightarrow |x_n - c| < \delta \).

\[ \underline{\text{Assume} \ n > n_0} \]

\[ |x_n - c| < \delta \Rightarrow |f(x_n) - f(c)| < \varepsilon \]

\[ \therefore \ \lim_{n \to \infty} f(x_n) = f(c) \]

\[ \implies \] Assume \( f \) has the property for all sequences \( (x_n)_{n \in \mathbb{N}} \) in \( D \) with \( \lim_{n \to \infty} x_n = c \) then \( \lim_{n \to \infty} f(x_n) = f(c) \).

(by contradiction)

Suppose \( f \) is not continuous at \( c \).

There exist an exceptional \( \varepsilon_0 > 0 \) so that for all choices of \( \delta > 0 \) there is \( x \in D \cap (c - \delta, c + \delta) \) and \( |f(x) - f(c)| \geq \varepsilon_0 \).

Let \( \delta_n = \frac{1}{n} > 0 \).

Choose \( x_1 \in D \cap (c - \frac{1}{n}, c + \frac{1}{n}) \)
\( x_2 \in D \cap (c - \frac{1}{n}, c + \frac{1}{n}) \)
\[ \vdots \]
\( x_n \in D \cap (c - \frac{1}{n}, c + \frac{1}{n}) \)
\[ \lim_{n \to \infty} x_n = c \quad \text{By the squeeze theorem} \]

But \( |f(x_n) - f(c)| \geq \varepsilon_0 \) for all \( n \).

\[ \lim_{n \to \infty} f(x_n) \neq f(c) \]

This contradicts the property \( f \) is supposed to have.

\[ \therefore f \text{ is continuous at } c \]

This is used to give us a couple conditions on continuity and discontinuity.

**Theorem (Sequential discontinuity)**

Let \( f : D \to \mathbb{R} \) and \( c \in D \)

1. If \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) are sequences in \( D \) with
   \[ \lim_{n \to \infty} x_n = c \quad \text{and} \quad \lim_{n \to \infty} y_n = c \quad \text{and} \quad \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n) \]
   then \( f \) is discontinuous at \( c \).

2. If \((x_n)_{n \in \mathbb{N}}\) is a sequence in \( D \) with \( \lim_{n \to \infty} x_n = c \)
   and \( \lim_{n \to \infty} f(x_n) \neq c \) then \( f \) is discontinuous at \( c \).

**Proof**

1. Limits of sequences are unique if \( f \) is continuous at \( c \) then \( \lim_{n \to \infty} f(x_n) = f(c) = \lim_{n \to \infty} f(y_n) \).

2. This is the contrapositive of the equivalence of sequential continuity theorem.

**Examples**

\[ f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \]

\( f \) is discontinuous at \( 0 \)

Let \( x_n = \frac{-1}{n} \) and \( y_n = \frac{1}{n} \)

\[ \lim_{n \to \infty} x_n = 0 \quad \text{and} \quad \lim_{n \to \infty} y_n = 0 \]

\[ \lim_{n \to \infty} f(x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} f(y_n) = 1 \]
$(q_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{N}$ to converge $J_{n_0} \in \mathbb{N}$.

So that $n > n_0 \Rightarrow q_{n_k} = q_0 \in \mathbb{N}$ (eventually constant)

$\Rightarrow x_{n_k} \rightarrow c$ and $\frac{p_{n_k}}{q_{n_k}} \rightarrow c$ and $\frac{p_{n_k}}{q_0} \rightarrow c$

$\Rightarrow p_{n_k} \rightarrow q_0 c \quad q_0 \in \mathbb{N}$

Since $(p_{n_k})$ is a sequence in $\mathbb{N}$ to converge $J_{n_0} \in \mathbb{N}$

So that $n > n_0 \Rightarrow p_{n_k} = p_0 \in \mathbb{N}$ (eventually constant)

$\Rightarrow \lim_{k \rightarrow \infty} p_{n_k} = p_0$

$\Rightarrow p_0 = q_0 c$

$\Rightarrow \frac{p_0}{q_0} = c$

$\Rightarrow c$ is natural (contradiction)
Let $f(x) = \begin{cases} 
0 & \text{x is rational} \\
1 & \text{x is irrational} 
\end{cases}$

$f(x)$ is discontinuous everywhere.

Let $c \in \mathbb{R}$.

Since $\mathbb{Q}$ is dense there exists $(x_n)_{n \in \mathbb{N}}$ for $\mathbb{Q}$ such that $x_n \to c$.

Since $\mathbb{R} \setminus \mathbb{Q}$ is dense there exists $(y_n)_{n \in \mathbb{N}}$ for $\mathbb{R} \setminus \mathbb{Q}$ such that $y_n \to c$.

But, $\lim_{n \to \infty} f(x_n) = 0$.

$\lim_{n \to \infty} f(y_n) = 1$.

$f(x) = \begin{cases} 
0 & \text{x is irrational} \\
\frac{1}{n} & \text{x is rational, x = $\frac{m}{n}$ with $\frac{m}{n}$ in lowest terms} 
\end{cases}$

$f(x)$ is continuous on the irrational numbers and discontinuous on the rational numbers.

To show $f$ is discontinuous on the rational numbers

Let $c \in \mathbb{Q}$.

There exists $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to c$.

But $f(c) = \frac{1}{n} \to 0$.

To show $f$ is continuous on the irrational numbers

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of rational numbers converging to $c \in \mathbb{R} \setminus \mathbb{Q}$.

$x_n = \frac{p_n}{q_n}$ with $\frac{p_n}{q_n}$ in lowest terms.

If $q_n$ bounded then there exist $(q_{n_k})$ a convergent subsequence.