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**Defn**
A sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence if for every \(\varepsilon > 0\), there is \(n_0 \in \mathbb{N}\) such that if \(m, n > n_0\), then \(|x_m - x_n| < \varepsilon\).

**Thm**
A convergent sequence is Cauchy.

**Alternate formulation**
If \((x_n)_{n \in \mathbb{N}}\) converges, then \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence.

**Proof**
Let \((x_n)_{n \in \mathbb{N}}\) be a convergent sequence with \(\lim_{n \to \infty} x_n = X\).

Let \(\varepsilon > 0\) be given.

\((x_n)\text{ convergent }\to\text{ There is } n_0 \text{ such that } n > n_0 \implies |x_n - X| < \frac{\varepsilon}{2}\)

Assume \(n, m > n_0\)

\[
|x_m - x_n| = |x_m - X + X - x_n| \\
\leq |x_m - X| + |X - x_n| \quad \text{(Triangle inequality)} \\
= |x_m - X| + |x_n - X| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon
\]

\((x_n)_{n \in \mathbb{N}}\text{ is Cauchy} \quad \blacksquare\)

**Thm** A Cauchy sequence is Bounded
If \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence, then \((x_n)_{n \in \mathbb{N}}\) is Bounded.

**Proof**
Assume \((x_n)_{n \in \mathbb{N}}\) is Cauchy.

Let \(\varepsilon = 1 > 0\)

\(\to\) There is \(n_0 \in \mathbb{N}\) such that \(m, n \geq n_0 \implies |x_m - x_n| < 1\)

Let \(B = \max \{ |x_1|, |x_2|, |x_3|, \ldots, |x_{n_0}|, 1 + |x_{n_0}| \} \)
Let $k \in \mathbb{N}$

**Case 1** \( k < n_0 \)

\[ |x_k| \leq \max \{ |x_1|, |x_2|, |x_3|, \ldots, |x_{n_0}|, 1 + |x_{n_0}| \} = B \]

**Case 2** \( k \geq n_0 \)

\[ |x_k| = |x_k - x_{n_0} + x_{n_0}| \]
\[ \leq |x_n - x_{n_0}| + |x_{n_0}| \quad \text{(triangle inequality)} \]
\[ < 1 + |x_{n_0}| \]
\[ \leq \max \{ |x_1|, |x_2|, |x_3|, \ldots, |x_{n_0}|, 1 + |x_{n_0}| \] = B

\[ |x_k| \leq B \] for all \( k \in \mathbb{N} \) \( \Rightarrow \)

\((x_n)_{n \in \mathbb{N}}\) is Bounded \( \blacksquare \)

**Theorem** A sequence converges if and only if it is a Cauchy sequence.

**Alt:** \((x_n)_{n \in \mathbb{N}}\) converges if and only if \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence.

**Proof**

\[ \Rightarrow \] Proof was given by theorem that a convergent sequence is Cauchy.

\[ \Leftarrow \] Assume \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence

\[ \Rightarrow (x_n)_{n \in \mathbb{N}} \] is Bounded

\[ \Rightarrow \] There exist \((x_{n_k})_{n \in \mathbb{N}}\) a convergent subsequence by the Bolzano-Weierstrass theorem for sequences

\[ \Rightarrow \] there is \( x_0 \in \mathbb{R} \) such that \( \lim_{k \to \infty} x_{n_k} = x_0 \)

(We want to show \( \lim_{n \to \infty} x_n = x_0 \))

Let \( \varepsilon > 0 \) be given

- \((x_n)_{n \in \mathbb{N}}\) Cauchy \( \Rightarrow \) there is \( n_1 \) such that \( n > n_1 \Rightarrow |x_n - x_0| < \frac{\varepsilon}{2} \)
- \((x_{n_k})_{n \in \mathbb{N}}\) converges \( \Rightarrow \) there is \( n_2 \) such that \( k > n_2 \Rightarrow |x_{n_k} - x_0| < \frac{\varepsilon}{2} \)

Let \( n_0 = \max \{ n_1, n_2 \} \)
Let $k > n_0 \quad k \in \mathbb{N}$

$\Rightarrow n_k \geq k > n_0$

$\Rightarrow |x_n - x_0| = |x_n - x_{n_k} + x_{n_k} - x_0|$

$\leq |x_n - x_{n_k}| + |x_{n_k} - x_0|$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\therefore \lim_{n \to \infty} x_n = x_0$

$(x_n)_{n \in \mathbb{N}}$ converges.

**Example**

$(x_n)_{n \in \mathbb{N}} = (\sqrt{n})_{n \in \mathbb{N}} = (1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots)$

is not Cauchy since it is not bounded.

But, $x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n}$ has a limit of zero!

$\lim_{n \to \infty} \sqrt{n+1} - \sqrt{n} = \lim_{n \to \infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \to \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$

To be Cauchy, it is important that indexes can be made arbitrarily far apart $|x_n - x_m| < \varepsilon$

not just by $|x_{n+1} - x_n|$

**Corollary**

If $(x_n)_{n \in \mathbb{N}}$ is a sequence such that $|x_{n+1} - x_n| \leq r |x_n - x_{n-1}|$ for all $n \in \mathbb{N}$

with $0 \leq r < 1$ then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

**proof (outline not full proof)**

1st $|x_{n+1} - x_n| \leq r |x_n - x_{n-1}| \leq r \cdot r |x_{n-1} - x_{n-2}| \leq r \cdot r \cdot r |x_{n-2} - x_{n-3}| \leq \ldots \leq r^n |x_2 - x_1|$

Or $|x_m - x_n| \leq r^{n-1} |x_2 - x_1|$

Assume $m > n$

$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \ldots + x_{n+1} - x_n|$

$\leq r^{m-2} |x_{m-1} - x_{n-1}| + r^{m-3} |x_{m-2} - x_{n-2}| + \ldots + r^n |x_2 - x_1|$

$\leq r^n |x_2 - x_1| \left( r \cdot r \cdot r \cdot \ldots + 1 \right)$

$= r^n |x_2 - x_1| \frac{1 - r^{m-n}}{1-r}$
\[
\frac{|x_n - x_1|}{r^n} \leq \frac{|x_n - x_1|}{1 - r}
\]

Since \( r < 1 \), \( \frac{|x_n - x_1|}{r^n} \to 0 \) as \( n \to \infty \).

For any \( \varepsilon > 0 \), there is \( n_0 \) such that \( n > n_0 \implies \frac{|x_n - x_1|}{r^n} < \frac{1 - r}{|x_2 - x_1|} \varepsilon \).

\[\implies \text{If } m > n > n_0 \text{ then } |x_m - x_n| < \varepsilon\]

\((x_n)_{n \in \mathbb{N}}\) is Cauchy.

**Example**

Let \((x_n)_{n \in \mathbb{N}}\) be given by \( x_n = \begin{cases} 
\alpha & n = 1 \\
0 & n = 2 \\
\frac{x_{n-1} + x_{n-2}}{2} & n > 2
\end{cases} \)

\[
x_1 = \alpha, \quad x_2 = 0, \quad x_3 = \frac{\alpha + 0}{2}, \quad x_4 = \frac{\alpha + \frac{\alpha + 0}{2}}{2} = \frac{\alpha + 3 \alpha}{4}, \quad x_5 = \frac{3 \alpha + 5 \alpha}{8}
\]

\[\therefore (x_n)_{n \in \mathbb{N}} \text{ Cauchy}
\]

\[\therefore (x_n)_{n \in \mathbb{N}} \text{ converges}
\]

What is \( \lim_{n \to \infty} x_n \)?

\[
x_n - x_1 = \sum_{k=2}^{n} x_k - x_{k-1} = \sum_{k=2}^{n} \left( -\frac{1}{2} \right)^{k-2} (x_k - x_1)
\]

\[= \left( \sum_{k=2}^{n} \left( -\frac{1}{2} \right)^{k-2} \right) (x_2 - x_1)
\]

\[= \left( \frac{1 - \left( -\frac{1}{2} \right)^{n-1}}{1 - \left( -\frac{1}{2} \right)} \right) (x_2 - x_1)
\]

\[= \left( 1 - \left( -\frac{1}{2} \right)^{n-1} \right) \frac{2}{3} (x_2 - x_1)
\]